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OPTIMUM CONTROL OF THE FLOW OF NON-NEWTONIAN FLUID BETWEEN ROTATING CYLINDERS

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An approximate method of solving the problem of optimum control of the motion of non-Newtonian fluid in the gap between rotating rollers is presented.

The production and reprocessing of many nonmetallic materials (such as plastic, rubber, heat- and sound-insulating materials, etc.) is carried out on rolling mills.

The hydrodynamic theory of rolling [1, 2] was successfully applied for determining the conditions of rolling mill operation. In that theory the motion of material in the gap between rollers is considered as the flow of a viscous non-Newtonian fluid, whose behavior at considerable rates of shear is satisfactorily described by the rheological power law [1 — 4].

As the result of complex shear strains a considerable part of mechanical energy is transformed into heat, thus considerably altering the flow temperature field.

The temperature of material in the gap is the determining factor in the production of quality products, and also one of the basic parameters in the calculation of the production process power requirements. Thermal processes determine the technological conditions for obtaining the required finish of rolled sheet, viscosity of the processed material, and the quality of the finished product [5]. An excessive temperature rise inside the sheet, induced by the intensive mechanical processing may result in the formation of cracks, bubbles, and foliation in sheets and finished products [6].

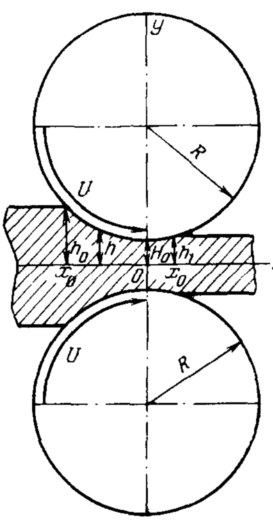

Fig. 1

The problem of maximizing the flow rate of a non-Newtonian fluid through the gap between two rotating cylinders (the problem of roller output maximization) is solved in this paper under conditions that would ensure the specified quality of products obtained at exit from the gap.

1. Let us consider the problem of motion of a non-Newtonian fluid in the gap between cylinders of radius R rotating in opposite directions at peripheral velocity U (Fig. 1). The minimum gap dimension $2H_0$ is assumed small in comparison with radius R $(R \gg 2H_0)$.

We assume that in the gap the flow is laminar, gravitational and inertial forces are small in comparison with friction forces and may be neglected, the hydrostatic pressure changes only in the direction of motion of the fluid, and that in the considered zone of fluid the material adheres to the cylinder surfaces. The motion of the fluid is symmetric about Ox. On these assumptions the equations of a plane nonisothermal flow of incompressible non-Newtonian fluid between rotating cylinders reduces to the system [7]

$$F_1 \equiv \frac{\partial p}{\partial x} - \frac{\partial \tau}{\partial y} = 0, \quad \frac{\partial p}{\partial y} = 0, \quad F_2 \equiv \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{1.1}$$

$$F_3 \equiv \tau - \mu_0 \exp\left[-b\left(\frac{T - T_0}{T_0 + 273}\right)\right]\left|\frac{\partial v_x}{\partial y}\right|^{n-1}\frac{\partial v_x}{\partial y} = 0$$

$$F_4 \equiv \rho c_p U_x \frac{\partial T}{\partial x} - \lambda \frac{\partial^2 T}{\partial y^2} - A\tau \frac{\partial v_x}{\partial y} = 0$$

with boundary conditions

$$\frac{\partial v_x}{\partial y} = \frac{\partial T}{\partial y} = 0, \quad v_y = 0 \quad \text{for} \quad y = 0 \tag{1.2}$$

(conditions of flow symmetry about Ox)

$$v_x = u\frac{R + H_0 - h}{R}, \quad v_y = U\frac{x}{R} \tag{1.3}$$

(conditions of fluid adhesion to cylinder walls), $T = T_w$ when $y = h$

$$p = 0, \quad T = T_0 \quad \text{for} \quad x = x_0 \tag{1.4}$$

$$p = 0, \quad dp/dx = 0 \quad \text{(conditions of separation)} \quad \text{for} \quad x = x_1 \tag{1.5}$$

In these formulas p is the pressure; τ is the tangential stress; v_x and v_y are components of flow velocity; T is the temperature distribution in the flow of fluid; μ_0 is the viscosity coefficient; n is the index of fluid flow; λ is the thermal conductivity coefficient; A is the mechanical equivalent of heat; R is the cylinder radius; T_0 is the temperature of fluid ahead of entry into the gap; T_w is the surface temperature of cylinders; b is a coefficient that defines the medium activation energy; x_0 and x_1 are the coordinates of the beginning and end of contact between medium and rollers; $h = R + H_0 - (R^2 - x^2)^{1/2}$ is the dimension of the half-gap.

The peripheral velocity U, the preheating temperature T_0 of the material, the surface temperature of rollers T_w, and the loading depth of the gap x_0 are controlling parameters of the process which satisfy the inequalities

$$\begin{gathered} 0 \leqslant U \leqslant U_{\max}, \quad T_{w\min} \leqslant T_w \leqslant T_{w\max} \\ T_{0\min} \leqslant T_0 \leqslant T_{0\max}, \quad x_{0\min} \leqslant x_0 \leqslant 0 \end{gathered} \tag{1.6}$$

We assume that for the specific values (U, T_0, T_w and x_0) there exists a unique solution of system (1.1) with conditions (1.2) — (1.5).

Among the admissible values of controlling parameters we have to find those that ensure the maximum flow rate of fluid through the gap at cross section x_1 (maximum output of rollers) or, what is equivalent, the minimum of functional

$$Q = -\int_0^{h_1} v_x(x_1, y)\, dy \tag{1.7}$$

with the isoperimetric relationship

$$\gamma = \int_{x_0}^{x_1}\int_0^{h} [T_j(x, y) - T(x, y)]^2\, dy\, dx = C \tag{1.8}$$

where the physical meaning of Q is that of flow rate, $T_j(x, y)$ is the specified temperature distribution in the gap region which is determined by technological conditions of rolling, γ is the magnitude of integral quadratic deviation of temperature $T(x, y)$ from the specified distribution $T_j(x, y)$, and C is a constant.

The method of solving variational problems is based on the universal device of variational calculus consisting of the inclusion of exact equations of motion and of the isoperimetric condition with undetermined multipliers in the supplementary functional (the method of Lagrange multipliers), and the calculations of variations of that functional [8].

2. The variation of parameters (U, T_0, T_w and x_0) alters the values of functions that define the state of the process. The variation of parameters and functions of state induce the variation of the flow region boundaries, i.e. the displacement of the boundary points x_0 and x_1.

We adjoin to system (1.1) the equation

$$F_5 \equiv q - \partial T / \partial y = 0$$

In conformity with the method proposed for solving variational problems of gasdynamics [8], the auxiliary functionals

$$Q^* = Q + \bar{Q}, \qquad \gamma^* = \gamma + \bar{\gamma}$$

are constituted and their first variations in the problem with movable boundaries x_0 and x_1: $\delta Q^* = \delta Q + \delta \bar{Q}$ and $\delta\gamma^* = \delta\gamma + \delta\bar{\gamma}$, where

$$\bar{Q} = \int_{x_1}^{x_0} \int_0^h \sum_{i=1}^{5} \lambda_i F_i \, dy \, dx$$

$$\bar{\gamma} = \int_{x_1}^{x_0} \int_0^h \sum_{i=1}^{5} \lambda_i^* F_i \, dy \, dx$$

are calculated. In the last formulas $\lambda_i = \lambda_i(x, y)$ and $\lambda_i^* = \lambda_i^*(x, y)$ $(i = 1, 2, \ldots, 5)$ are the undetermined multipliers.

In the expressions of variations δQ^* and $\delta\gamma^*$ we retain only variations of controlling parameters that are independent, and exclude remaining variations.

For this we select the multipliers $\lambda_i = \lambda_i(x, y)$ and $\lambda_i^* = \lambda_i^*(x, y)$ so that the expressions at variations of phase coordinates in δQ^* and $\delta\gamma^*$ vanish. This yields two systems of equations in λ_i and λ_i^*

$$\frac{\partial \lambda_2}{\partial x} - n \frac{\partial}{\partial y}\left(\tau\lambda_3 \Big/ \frac{\partial v_x}{\partial y}\right) - \rho c_p \lambda_4 \frac{\partial T}{\partial x} - A \frac{\partial}{\partial y}(\tau\lambda_4) = 0 \qquad (2.1)$$

$$\lambda_3 = A \frac{\partial v_x}{\partial y} \lambda_4 - \frac{\partial \lambda_1}{\partial y}$$

$$\rho c_p \frac{\partial}{\partial x}(\lambda_4 v_x) - \frac{\partial \lambda_5}{\partial y} - \frac{b\tau}{T_0 + 273} \lambda_3 = 0, \quad \lambda_5 = -\lambda \frac{\partial \lambda_4}{\partial y}$$

$$\frac{d}{dx} \int_0^h \lambda_1 \, dy = 0, \quad \frac{\partial \lambda_2}{\partial y} = 0$$

$$\frac{\partial \lambda_2^*}{\partial x} - n \frac{\partial}{\partial y}\left(\tau\lambda_3^* \Big/ \frac{\partial v_x}{\partial y}\right) - \rho c_p \lambda_4^* \frac{\partial T}{\partial x} - A \frac{\partial}{\partial y}(\tau\lambda_4^*) = 0 \qquad (2.2)$$

$$\lambda_3{}^* = A\frac{\partial v_x}{\partial y}\lambda_4{}^* - \frac{\partial \lambda_1{}^*}{\partial y}$$

$$\rho c_p \frac{\partial}{\partial x}(\lambda_4{}^* v_x) - \frac{\partial \lambda_5{}^*}{\partial y} - \frac{b\tau}{T_0+273}\lambda_3{}^* + 2(T_j - T) = 0$$

$$\lambda_5{}^* = -\lambda\frac{\partial \lambda_4{}^*}{\partial y}, \quad \frac{d}{dx}\int_0^h \lambda_1{}^* dy = 0, \quad \frac{\partial \lambda_2{}^*}{\partial y} = 0$$

Boundary conditions for the adjoint systems are determined by the condition that the coefficients at variations of those of functions δQ^* and $\delta\gamma^*$ whose values at the boundary are not specified, as well as at variation of the mobile point x_1, must vanish. We have

$$\begin{aligned}
&\lambda_3 = \lambda_5 = 0, \quad \lambda_3{}^* = \lambda_5{}^* = 0 \quad \text{for} \quad y = 0 \\
&\lambda_1 = \lambda_4 = 0, \quad \lambda_1{}^* = \lambda_4{}^* = 0 \quad \text{for} \quad y = h \\
&\lambda_2 = 0, \quad \lambda_2{}^* = 0 \quad \text{for} \quad x = x_0 \\
&\lambda_4 = 0, \ \lambda_4{}^* = 0, \quad \int_0^h \left(\lambda_1\frac{\partial \tau}{\partial y} - \lambda_2\frac{\partial v_y}{\partial y}\right)dy = v_x\frac{dh}{dx} \\
&\int_0^h \left[\lambda_1{}^*\frac{\partial \tau}{\partial y} - \lambda_2{}^*\frac{\partial v_y}{\partial y} - (T_j - T)^2\right]dy = 0 \quad \text{for} \quad x = x_1
\end{aligned} \tag{2.3}$$

Variations δQ^* and $\delta\gamma^*$ with allowance for (2.1)—(2.3) are defined by

$$\begin{aligned}
\delta Q^* = &-\left[\int_{x_1}^{x_0}\left(n\frac{R+H_0-h}{R}\tau\lambda_3 \Big/ \frac{\partial v_x}{\partial y}\right)_{y=h} dx + \int_0^{h_1}(\lambda_2+1)_{x=x_1} \times \right. \\
&\left.\frac{R+H_0-h_1}{R}dy\right]\delta U + \left[\int_0^{h_0}(\rho c_p\lambda_4 v_x)_{x=x_0} dy - \right. \\
&\left.\int_{x_1}^{x_0}\int_0^h b\frac{T+273}{(T_0+273)^2}\tau\lambda_3\, dy\, dx\right]\delta T_0 - \int_{x_1}^{x_0}\lambda_5(x,h)\, dx\delta T_w - \\
&\int_0^{h_0}\left[\lambda_1\frac{\partial \tau}{\partial y} + A\tau\lambda_4\frac{\partial v_x}{\partial y}\right]_{x=x_0} dy\delta x_0
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
\delta\gamma^* = &-\left[\int_{x_1}^{x_0}\left(n\frac{R+H_0-h}{R}\tau\lambda_3{}^* \Big/ \frac{\partial v_x}{\partial y}\right)_{y=h} dx + \right. \\
&\left.\int_0^{h_1}\lambda_2{}^*(x_1,y)\frac{R+H_0-h_1}{R}dy\right]\delta U + \left[\int_0^{h_0}(\rho c_p\lambda_4{}^* v_x)_{x=x_0} dy - \right. \\
&\left.\int_{x_1}^{x_0}\int_0^h b\frac{T+273}{(T_0+273)^2}\tau\lambda_3{}^*\, dy\, dx\right]\delta T_0 - \int_{x_1}^{x_0}\lambda_5{}^*(x,h)\, dx\delta T_w - \\
&\int_0^{h_0}\left[\lambda_1{}^*\frac{\partial \tau}{\partial y} + A\tau\lambda_4{}^*\frac{\partial v_x}{\partial y} - (T_j - T)^2\right]_{x=x_0} dy\delta x_0
\end{aligned} \tag{2.5}$$

3. One of the methods of solving variational problems on conditional extremum is that of sequential descent [9]. This method is a generalization of the basic ideas of the method of steepest descent and is extended to systems with distributed parameters and isoperimetric links.

We specify the first approximation of controlling parameters ($U^{(1)}$, $T_0^{(1)}$, $T_w^{(1)}$ and $x_0^{(1)}$) in conformity with that method. There are no general recommendations for constructing the first approximation. A successive selection may ensure the minimum of functional Q with isoperimetric condition $\gamma = C$ and differential relationships (1.1). To satisfy all these conditions in the first approximation is not always possible.

We assume that the controlling parameters satisfy in what follows the inequalities (1.6). If the obtained values exceed the limits admitted by (1.6), we use the limit values of controlling parameters.

Using the first approximation we construct the solution of system (1.1) — (1.5) and determine the values of functionals Q and γ. It should be expected that $\gamma \neq C$ and $Q \neq \min Q$. Then on the basis of the first approximation of phase coordinates we determine the undetermined multipliers λ_i and λ_i^* $(i = 1, 2 \ldots 5)$.

Applying the method of sequential descent we achieve the fulfillment of conditions $\gamma = C$ and $Q = \min Q$. First we fix the value of functional Q^*, and assume that its first variation in the second and subsequent approximations

$$\delta Q^{*(2)} = \delta Q^{*(3)} = \ldots = \delta Q^{*(n)} = 0 \tag{3.1}$$

is zero.

To satisfy the condition $\gamma = C$ we constitute with the use of formulas (2.4) and (2.5) the functional

$$\delta\gamma = \delta\gamma^* + \alpha\delta Q^* \tag{3.2}$$

where α is a constant multiplier.

From (3.2), by the method of steepest descent, we obtain variations of the controlling parameters

$$\delta U = \varepsilon_1 \left\{ \int_{x_1}^{x_2} \left[n\tau \left(\frac{\partial v_x}{\partial y} \right)^{-1} (\lambda_3^* + \alpha\lambda_3) \right]_{y=h} dx + \int_0^{h_1} \frac{R + H_0 - h_1}{R} [\lambda_2^* + \alpha(\lambda_2 + 1)]_{x=x_1} dy \right\} \tag{3.3}$$

$$\delta T_0 = -\varepsilon_2 \left\{ \int_0^{h_2} \rho c_p v_x (\lambda_4^* + \alpha\lambda_4)|_{x=x_2} dy - \int_{x_1}^{x_2}\int_0^{h} b \frac{T + 273}{(T_0 + 273)^2} \tau (\lambda_3^* + \alpha\lambda_3)\, dy\, dx \right\}$$

$$\delta T_w = \varepsilon_3 \int_{x_1}^{x_2} (\lambda_5^* + \alpha\lambda_5)_{y=h}\, dx$$

$$\delta x_0 = \varepsilon_4 \int_0^{h_2} \left[\frac{\partial\tau}{\partial y} (\lambda_1^* + \alpha\lambda_1) + A\tau \frac{\partial v_x}{\partial y} (\lambda_4^* + \alpha\lambda_4) - (T_j - T)^2 \right]_{x=x_2} dy$$

where the constant ε_i $(i = 1, 2, 3, 4)$ determines the approximation step.

We substitute variations δU, δT_0, δT_w and δx_0 into the formula for δQ^* (2.4) and, setting in accordance with (3.1) $\delta Q^* = 0$, obtain the equation for the determination of the constant multiplier α.

Using small ε_i $(i = 1, 2, 3\ 4)$ we carry out several approximations for achieving $\gamma = C$, as is made in the method of steepest descent. To obtain in each approximation the values of $(U^{(s)} + \delta U^{(s+1)},\ T_0^{(s)} + \delta T_0^{(s+1)},\ T_w^{(s)} + \delta T^{(s+1)}$ and $x_0^{(s)} + \delta x_0^{(s+1)})$, where s is the number of approximation, we solve the system of Eqs. (1.1), (2.1) and (2.2), determine its multiplier α in (3.2), and specify its step ε_i.

Having obtained the fulfillment of the isoperimetric condition $\gamma = C$, we pass to the derivation of controlling parameters that ensure $\min Q$. The derivation procedure is the same as used in the derivation of parameters which satisfy the specified condition $\gamma = C$. We fix the value of functional γ^* and constitute $\delta Q = \delta Q^* + \alpha\delta\gamma^*$, then, using the method of steepest descent, determine variations of parameters $\delta U, \delta T_0, \delta T_w$ and δx_0. Setting in each step $\delta\gamma^* = 0$, we determine the constant α, and so on.

The problem is considered solved when the condition $Q = \min Q$ is reasonably exactly satisfied.

The actual derivation of the optimal solution requires the ability to construct solutions of equations of motion with an arbitrary law of variation of controlling parameters and of adjoint systems with known coefficients in their equations.

4. The solution of equations of motion (1.1) — (1.5) is given in [7].

In dimensionless coordinates $\xi = x/\sqrt{2RH_0}$ and $\eta = y/h$ the basic parameters of flow are of the form

$$v_\xi = U\left\{(R + H_0 - h - h_1)(h_1 - h)\left[Rh\int_0^1\int_1^\eta\left(\frac{\varphi}{\psi}\right)^{1/n} d\varphi\, d\eta\right]^{-1} \times \right. \tag{4.1}$$

$$\left. \int_1^\eta\left(\frac{\varphi}{\psi}\right)^{1/n} d\varphi + \frac{R + H_0 - h}{R}\right\}$$

$$v_\eta = -h\int_0^\eta \frac{\partial v_\xi}{\partial \xi}\, d\varphi$$

$$p(\xi) = \mu_0\left(\frac{U}{R}\right)^n\int_{\xi_0}^{\xi}\left\{(R + H_0 - h - h_1) \times \right.$$

$$\left.\left[h^{1+2n/n}\int_0^1\int_1^\eta\left(\frac{\varphi}{\psi}\right)^{1/n} d\varphi\, d\eta\right]^{-1}\right\}^n |h_1 - h|^{n-1}(h_1 - h)\, dx$$

where

$$\psi = \exp\left[-b\left(\frac{T - T_0}{T_0 + 273}\right)\right], \quad h = R + H_0 - (R^2 - \xi^2)^{1/2}$$

$$h_1 = R + H_0 - (R^2 - \xi_1^2)^{1/2}, \quad \xi_1 = \frac{x_1}{\sqrt{2RH_0}}$$

Parameter h_1 or coordinate ξ_1 are determined by the condition $p(\xi_1) = 0$. When the latter is satisfied we have

$$\int_{\xi_0}^{\xi_1}\left\{[(R^2 - \xi^2)^{1/2} + (R^2 - \xi_1^2)^{1/2} - R - H_0] \times \right.$$

$$\left[h^{1+2n/n}\int_0^1\int_1^{\eta}\left(\frac{\varphi}{\psi}\right)^{1/n} d\varphi\, d\eta\right]^{-1}\Big\}^n \left|(R^2-\xi^2)^{1/2}-(R^2-\xi_1^2)^{1/2}\right|^{n-1}\times$$
$$[(R^2-\xi^2)^{1/2}-(R^2-\xi_1^2)^{1/2}]\,d\xi = 0$$

Introducing the dimensionless temperature difference

$$\theta = T - T_0/T_0$$

we reduce the equation of energy in coordinates ξ and η to the form

$$\frac{\partial\theta}{\partial\xi} = \frac{\nu}{\sigma}\,\frac{\sqrt{2RH_0}}{h^2 v_\xi}\,\frac{\partial^2\theta}{\partial\eta^2} + \frac{B(\xi)}{v_\xi}\exp(-b_0\theta)\left|\frac{\partial v_\xi}{\partial\eta}\right|^{n+1} \tag{4.2}$$

where σ is the Prandtl number, ν is the kinematic voscosity coefficient, and

$$B(\xi) = \frac{A\mu_0\sqrt{2RH_0}}{\rho c_p T_0 h^{n+1}},\quad b_0 = b\,\frac{T_0}{T_0+273}$$

We seek the temperature distribution in the form of series

$$\theta = \theta_{00} + \sum_{k=1}^{\infty}\theta_k(\xi)\cos\frac{k\pi}{2}\eta,\quad k = 1, 3, 5, \ldots . \tag{4.3}$$

Substituting (4.3) into Eq. (4.2), multiplying both sides of that inequality by $\cos m\pi/2\eta$, where m also admits odd values, and integrating with respect to η in the interval(0, 1), for the determination of $\theta_k(\xi)$ we obtain the system of equations

$$\frac{d\theta_m}{d\xi} + \sum_{k=1}^{\infty}A_{km}\theta_k + C_m = 0;\quad m = 1, 3, 5, . \tag{4.4}$$

The coefficients in Eqs. (4.4) are of the form

$$A_{km} = \frac{k^2\pi^2\nu\sqrt{2RH_0}}{2\sigma h^2}\int_0^1 v_\xi^{-1}\cos\frac{k\pi}{2}\eta\cos\frac{m\pi}{2}\eta\, d\eta$$

$$C_m = -2B(\xi)\int_0^1 v_\xi^{-1}\left|\frac{\partial v_\xi}{\partial\eta}\right|^{n+1}\exp(-b_0\theta)\cos\frac{m\pi}{2}\eta\, d\eta$$

The system of Eqs. (4.4) is solved for boundary condition $\theta_k(\xi_0) = 0$.

The derived formulas were used for the numerical computation of physical flow parameters of the fluid in the gap between cylinders of radius $R = 0.25$ m. The maximum gap $2H_0 = 0.2\cdot 10^{-3}$ m, the depth of loading $\xi_0 = -2$, the rollers surface temperature $T_w = T_0 = 150°$ C, and the peripheral velocity of rollers $U = 0.8$ m/sec. The thermophysical properties of the fluid are defined by the following parameters: $\mu_0 = 0.484\cdot 10^5$ N $\cdot$ sec/m^2; $n = 0.23$; $\rho = 1.38$ N/m^3; $b = 26.6$; $c_p = 0.025$ kcal/N $\cdot$ deg; $\lambda = 0.1\cdot 10^{-3}$ kcal/m $\cdot$ sec $\cdot$ deg.

Derivation of the process functions of state is carried out by the method of successive approximations.

In the zero approximation $\theta^{(0)}(\xi, \eta) = 0$ is assumed. Formula (4.1) is used for calculating on the basis of the zero approximation of temperature the zero approximation $v_\xi^{(0)}, v_\eta^{(0)}, p^{(0)}(\xi)$, while $A_{km}^{(0)}$ and $C_m^{(0)}$ are determined by the zero approximation $v_\xi^{(0)}$. These approximations are used for solving system (4.4), then the first approximation of temperature $\theta^{(1)}(\xi, \eta)$ (4.3) is determined, and so on.

Convergence of successive approximations is not considered here, but in the example of calculations presented here it was found that the second approximation of the function of state differed from the first one by not more than 3%.

The distribution of velocities v_ξ and v_η, and of the pressure $p(\xi)$ in the zero (solid lines) and second (dash lines) approximations are shown in Figs. 2 — 4. The second approximation of temperature at cross section $\xi = -1.8$ represented by the sum of three terms of series (4.3) and computed with the use of the zero approximation of v_ξ appears in Fig. 5 (owing to its symmetry only the region $0 \leqslant \eta \leqslant 1$ is shown).

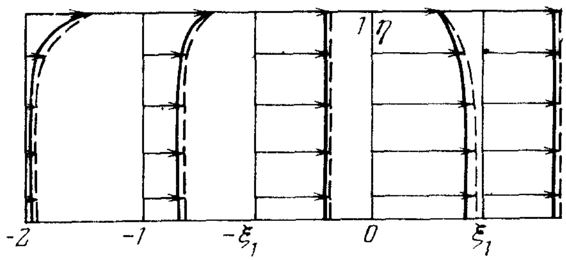

Fig. 2

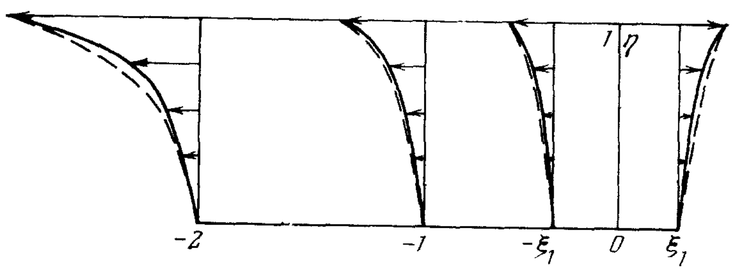

Fig. 3

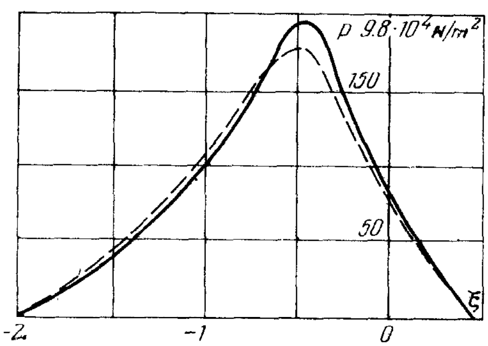

Fig. 4

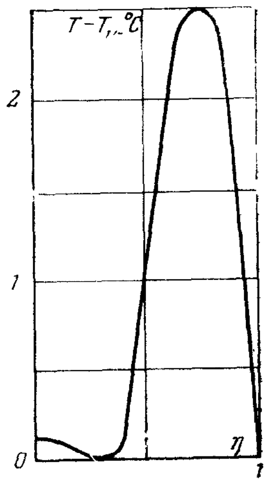

Fig. 5

Let us consider the integration of the adjoint system of equations, and obtain the solution of system (2. 2). We eliminate λ_3^* and λ_5^* from the equations,and represent these in dimensionless coordinates ξ and η as

$$\frac{d\lambda_2^*}{d\xi}+b(\xi)\psi\left|\frac{\partial v_\xi}{\partial\eta}\right|^{n-1}\frac{\partial^2\lambda_1^*}{\partial\eta^2}+b(\xi)\frac{\partial}{\partial\eta}\left(\psi\left|\frac{\partial v_\xi}{\partial\eta}\right|^{n-1}\right)\frac{\partial\lambda_1^*}{\partial\eta}-\tag{4.5}$$

$$\rho c_p\lambda_4^*\frac{\partial T}{\partial\xi}-A\frac{n+1}{h}\frac{\partial}{\partial\eta}(\tau\lambda_4^*)|=0$$

$$\frac{\partial\lambda_4^*}{\partial\xi}+\frac{a(\xi)}{v_\xi}\frac{\partial^2\lambda_4^*}{\partial\eta^2}-\left(v_\xi^{-1}\frac{\partial v_\xi}{\partial\xi}-\frac{AB\tau}{\sqrt{2RH_0}hv_\xi}\frac{\partial v_\xi}{\partial\eta}\right)\lambda_4^*-$$

$$\frac{B\tau}{\sqrt{2RH_0}hv_\xi}\frac{\partial\lambda_1^*}{\partial\eta}+\frac{2\sqrt{2RH_0}(T_j-T)}{\rho c_p v_\xi}=0$$

$$\int_0^1\lambda_1^*d\eta=\frac{\text{const}}{h}\tag{4.6}$$

The dimensions of multipliers λ_i^* (i = 1, 2, 4) are: $[\lambda_1^*]=\text{deg}^2\text{m}^3/\text{N}$, $[\lambda_2^*]=\text{deg}^2$ sec, $[\lambda_4^*]=\text{deg}^2\text{m}^3\text{sec/kcal}$. After these transformations the boundary conditions assume the form

$$\frac{\partial\lambda_1^*}{\partial\eta}=\frac{\partial\lambda_4^*}{\partial\eta}=0\quad\text{for}\quad\eta|=0\tag{4.7}$$

$$\lambda_1^*=\lambda_4^*=0\qquad\qquad\text{for}\quad\eta=1$$

$$\lambda_2^*=0\quad\text{for}\quad\xi=\xi_0$$

$$\lambda_4^*=0,\quad\lambda_2^*=-Rh_1[U\xi_1]^{-1}\int_0^1(T_j-T)^2\,d\eta\quad\text{for}\quad\xi=\xi_1$$

where

$$b(\xi)=\frac{n\mu_0}{\sqrt{(2RH_0)^n}\,h^{n+1}},\quad a(\xi)=\frac{\nu}{\sigma h\sqrt{2RH_0}}$$

$$B=\frac{b\sqrt{2RH_0}}{\rho c_p(T_0+273)},\quad h=h(\xi)$$

Multipliers λ_1^* and λ_4^* are sought in the form

$$\lambda_1^*(\xi,\eta)=\sum_{m=1}^{\infty}\lambda_{1m}^*(\xi)|\cos\frac{m\pi}{2}\eta,\quad\lambda_4^*(\xi,\eta)=\sum_{k=1}^{\infty}\lambda_{4k}^*(\xi)\cos\frac{k\pi}{2}\eta\tag{4.8}$$

which satisfies boundary conditiond (4. 7). Here m and k are odd.

As in the solution of the energy equation, we substitute expansions (4. 8) into Eqs. (4.5) multiplying these, respectively, by $\cos s\pi/2\ \eta$ and $\cos p\pi/2\ \eta$ (s and p are odd), integrate with respect to η in the interval (0, 1), and obtain

$$\frac{d\lambda_2^*}{d\xi}-\sum_{m=1}B_{ms}\lambda_{1m}^*-\sum_{k=1}^{\infty}D_{ks}\lambda_{4k}^*=0\tag{4.9}$$

$$\frac{d\lambda_{4p}^*}{d\xi}-\sum_{k=1}^{\infty}A_{kp}\lambda_{4k}^*-\sum_{m=1}^{\infty}C_{mp}\lambda_{1m}^*+C_p^*=0$$

where

$$B_{ms} = b(\xi)\frac{s\pi}{2\sin s\pi/2}\int_0^1\left[\frac{m^2\pi^2}{4}\psi\left|\frac{\partial v_\xi}{\partial\eta}\right|^{n-1}\cos\frac{m\pi}{2}\eta +\right.$$
$$\left.\frac{m\pi}{2}\frac{\partial}{\partial\eta}\left(\psi\left|\frac{\partial v_\xi}{\partial\eta}\right|^{n-1}\right)\sin\frac{m\pi}{2}\eta\right]\cos\frac{s\pi}{2}\eta\,d\eta$$
$$D_{ks} = \frac{s\pi}{2\sin s\pi/2}\int_0^1\left[\left(\rho c_p\frac{\partial T}{\partial\xi} + \frac{A(n+1)}{h}\frac{\partial\tau}{\partial\eta}\right)\cos\frac{k\pi}{2}\eta -\right.$$
$$\left.\frac{k\pi}{2}A\frac{n+1}{h}\tau\sin\frac{k\pi}{2}\eta\right]\cos\frac{s\pi}{2}\eta\,d\eta$$
$$A_{kp} = 2\int_0^1\left[\frac{k^2\pi^2 a(\xi)}{4v_\xi} - \left(v_\xi^{-1}\frac{\partial v_\xi}{\partial\xi} - A\frac{B\tau}{\sqrt{2RH_0}hv_\xi}\frac{\partial v_\xi}{\partial\eta}\right)\right]\cos\times$$
$$\frac{k\pi}{2}\eta\cos\frac{p\pi}{2}\eta\,d\eta$$
$$C_{mp} = \frac{2B}{h\sqrt{2RH_0}}\int_0^1\frac{m\pi\tau}{2v_\xi}\sin\frac{m\pi}{2}\eta\cos\frac{p\pi}{2}\eta\,d\eta$$
$$C_p^* = \frac{4\sqrt{2RH_0}}{\rho c_p}\int_0^1\frac{T_j - T}{v_\xi}\cos\frac{p\pi}{2}\eta\,d\eta$$

The integral condition (4.6) after the substitution into it of expansion of λ_1^* from(4.8) and transformations, assumes the form

$$\sum_{m=1}^{\infty}\frac{\lambda_{1m}^*}{m}\sin\frac{m\pi}{2} = \frac{\text{const}}{h(\xi)} \qquad (4.10)$$

The boundary conditions for the system (4.9),(4.10) are:

$$\lambda_2^*(\xi_0) = 0, \quad \lambda_{4k}^*(\xi_1) = 0$$

For determining the constant we use the value of function λ_2^* at the boundary $\xi=\xi_1$

Integration of Eqs. (4.9) and (4.10) yields the distribution of $\lambda_2^*(\xi)$, $\lambda_{1m}^*(\xi)$ and $\lambda_{4k}^*(\xi)$ $(m, k = 1,3,5,\ldots)$, which together with expansions (4.8) represent the solution of system (4.5).

Solution of system (2.1) is similarly derived.

Let us consider the example of computing the optimal control. Let $T_j = \text{const} = 150°\,C$, and $\gamma = C = 2\ \text{deg}^{-2}$.

We shall determine the optimal peripheral velocity U_0, wall temperature T_{w0}, temperature T_{00} of preheating of the material, and the optimal depth ξ_{00} of roller loading during the processing of a material whose thermophysical properties are described above.

We specify the controlling parameters in the first approximation with allowance for the inequalities $0 \leqslant U \leqslant U_{\max} = 0.8$ m/sec, $\quad 0 \leqslant T_w \leqslant T_{w\max} = 150°\,C$

$$0 \leqslant T_0 \leqslant T_{0\max} = 170°\,C, \quad 0 \geqslant \xi_0 \geqslant \xi_{0\min} = -2$$

Let

$$U^{(1)} = U_{\max} = 0.8\ \text{m/sec},\ T_0^{(1)} = 155°\,C;\ T_w^{(1)} = 145°\,C,\ \xi_0 = -2$$

The distribution of temperature θ, and functions λ_i and λ_i^* $(i = 1, 2, 4)$ will be represented by a single term of the expansion.

We solve the input system of Eqs. (1.1) — (1.5) with the use of the first approximation of parameters $(U^{(1)}, T_0^{(1)}, T_w^{(1)}$ and $\xi_0^{(1)})$ and check the fulfillment of conditions $Q = \min Q$ and $\gamma = C$. In the considered case $Q = \min Q$, but $\gamma > C$.

We determine multipliers λ_i and λ_i^* $(i = 1, 2, 4)$, and constitute the expressions of parameter variations and the equations for determining the constant multiplier α. We specify the following approximation steps: $\varepsilon_1 = 0.0005$; $\varepsilon_2 = 29$; $\varepsilon_3 = 1.25$ and $\varepsilon_4 = 0.0005$ and determine α.

We determine the increments $\delta U, \delta T_0, \delta T_w$ and δx_0 and the second approximation for the control parameters $(U^{(2)}, T_0^{(2)}, T_w^{(2)}$ and $\xi_0^{(2)})$.

In the second approximation coordinate $\xi_0^{(2)}$ exceeds the admissible limit $\xi_0 = -2$. In subsequent approximations we set $\xi_0 = -2$. We achieve the fulfillment of condition $\gamma = C$ in the third approximation, which means that $U^{(3)} = U_0$, $T_0^{(3)} = T_{00}$, $T_w^{(3)} = T_{w0}$, and $\xi_0^{(3)} = \xi_{00}$ represent optimal values. Results of computations are given in Table 1.

Table 1

approximation №	U, m/sec	T_0, deg	T_w, deg	ξ_0	$Q \cdot 10^{-6}$, m²/sec	γ, deg²
1	0.8	155	145	—2	94.75	23.45
2	0.755	150	146.1	—2	89.60	14.11
3	0.704	144.2	147.9	—2	83.70	2.02

Table 2

approximation №	U, m/sec	T_0, deg	T_w, deg	ξ_0	$Q \cdot 10^{-6}$, m²/sec	γ, deg²
1	0.704	144.2	147.9	—2	83.70	6.887
2	0.606	134.8	150	—2	72.10	2.43
3	0.515	129.5	150	—2	60.92	2.38
4	0.400	124	150	—2	47.70	2.0

Results of computations of optimal control presented in Table 2 relate to the case in which the the distribution of temperature θ and multipliers λ_i and λ_i^* $(i = 1, 2, 4)$ are defined by the sum of the first two terms of the series.

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